Section B5 – Individual Project IP5-PL

Individual project's contribution to the CRP

The Polish team is led by Jarosław Grytczuk at the Jagiellonian University in Kraków. The other members are Tomasz Krawczyk and Piotr Micek. The group is completed by three PhD students and three Master students. The team is responsible for work package WP09 – Coloring graphs with geometric representations, which is described below. The team members will also contribute to work packages WP06 and WP10.

WP09 – Coloring graphs with geometric representations Coloring problems constitute the core of combinatorics from the very beginning. They are fascinating for many reasons, ranging from purely aesthetic to practical ones. Coloring problems for geometric structures are most exciting in both respects. Indeed, on one hand we have grand challenges (like determining the chromatic number of the plane), on the other, a variety of topics inspired by real-world applications (like dynamic storage allocation, frequency assignment, VLSI design, etc.).

In this project we are mainly interested in coloring intersection graphs of intervals and their two-dimensional relatives (rectangles, disks, etc.). In a typical problem we are given a family of objects whose members are to be colored so that the intersecting ones get different colors. The task is to find an effective coloring procedure using as few colors as possible.

Basically we distinguish two settings: static (off-line), when the entire structure is known in advance, and dynamic (on-line), when the objects are coming one by one and should be colored immediately and irrevocably. The latter model corresponds to many real-world applications in which the input is gradually disclosed over time. As an intermediate variant we study coloring games in which the structure is given in advance, but only one of the players cares about minimizing the number of colors. We also plan to investigate other types of colorings in this geometrical vein (e.g. game colorings, nonrepetitive colorings, etc.).

One direction of our research focuses on a general question: how much beneficial it is to use geometric representation, rather than just an abstract graph, in a coloring algorithm? For instance, it is known that in the on-line coloring of unit interval graphs one needs at least $2\omega - 1$ colors in the abstract case (which is best possible [6]), while currently best lower bound in the geometric case is $\frac{3}{2}\omega$ (where ω denotes the clique number). We aim at closing this gap. Similar dissonance is expected for bounded tolerance graphs. Since these graphs are complements of comparability graphs [7], we may use here on-line chain partitioning algorithms for posets [4]. By the recent progress in this area made by Bosek and Krawczyk [5] we know that $O(\omega^{16\log\omega})$ colors are sufficient, while the current lower bound is $\binom{\omega+1}{2}$. We feel however that making use of geometric representation one can get here an on-line algorithm using the number of colors polynomial in ω. Recently it was proved that the first-fit algorithm uses $\Theta(\omega)$ colors on p-tolerance graphs [8], which are a subclass of bounded tolerance graphs.

There are many ways of generalizing intervals from the real line to higher dimensions. For most of them the coloring problems become much harder even in the off-line setting. Consider for instance the class of intersection graphs of line segments in the plane. It is not known if the chromatic number of these graphs is bounded in terms of the clique number (cf. [10]). The situation is not so hopeless for box graphs (intersection graphs of axis-parallel rectangles), though there is no progress on the half-century-old upper bound of $O(\omega^2)$ due to Asplund and Grünbaum [2] (and nothing better than $\Omega(\omega)$ is known from below). Restricting to translated or homothetic copies of one particular figure allows for linear upper bounds on the chromatic number ($3\omega - 2$ in the former case, and $6\omega - 6$ in the later), as proved in [9]. Moreover, these bounds are valid in a stronger sense, where the chromatic number is replaced by the coloring number (the smallest k for which a graph is $(k-1)$-degenerate). These bounds are known to be sharp only for unit disks. We aim at improving them for other classes, like general disk graphs, square graphs, etc. We also hope for obtaining (close to) optimal on-line algorithms for some of these classes, including general box graphs.

Aside from attacking classical problems, we would like to introduce some new topics into the area. One of them is the graph coloring game [3], where two players color a given graph (or its geometric representation) together, but only one of them is interested in minimizing the number of colors used (the other one tries to maximize it). The value of the game when both players play perfectly is known as the game chromatic number. The game was originally proposed for planar graphs, and, despite extensive efforts, the exact value remains a mystery (for some time it was not even clear if it is finite). We hope to achieve some progress here, perhaps by using appropriate geometric representation of planar graphs.

A related concept that recently gained considerable attention is nonrepetitive graph coloring [1]. In such coloring repetitive paths (of any length) are forbidden (a path is repetitive if its first half looks exactly like the second half). A major challenge in this topic is to decide whether every planar graph is nonrepetitively k-colorable for some absolute constant k.

Milestones of WP09

- M09.1 Improve the bounds for the on-line approximability of the chromatic number for unit interval graphs.
- M09.2 Try to provide a polynomial on-line approximation of the chromatic number for bounded tolerance graphs; analyze the possibilities of on-line approximation for general tolerance graphs.
- M09.3 Provide/improve upper bounds for the chromatic/coloring number in terms of the clique number for various classes of intersection graphs.
- M09.4 Provide a reasonable on-line coloring algorithm for box graphs.
- M09.5 Improve the bounds for the game chromatic number of planar graphs.
- M09.6 Verify whether the nonrepetitive chromatic number of planar graphs is finite.

References for WP09

[1] N. Alon, **J. Grytczuk**, M. Hałuszczak, and O. Riordan. Nonrepetitive colorings of graphs. *Random Struct. Algorithms*, 21(3–4):336–346, 2002.

[2] E. Asplund and B. Grünbaum. On a coloring problem. *Math. Scand.*, 8:181–188, 1960.

[3] T. Bartnicki, J. Grytczuk, H. A. Kierstead, and X. Zhu. The map coloring game. *Amer. Math. Monthly*, 114:793–803, 2007.

[4] B. Bosek, **S. Felsner**, K. Kloch, **T. Krawczyk**, G. Matecki, and **P. Micek**. On-line chain partitions of orders: a survey. Submitted.

[5] B. Bosek and **T. Krawczyk**. A subexponential upper bound for on-line chain partitioning problem. Manuscript.

[6] **S. Felsner**, K. Kloch, G. Matecki, and **P. Micek**. On-line chain partitions of up-growing semi-orders. Submitted.

[7] M. C. Golumbic and C. L. Monma. A generalization of interval graphs with tolerances. In: *Proc. Southeastern Conf. Combinatorics, Graph Theory & Comput.*, vol. 35, p. 321–331, 1982.

[8] H. A. Kierstead and K. R. Saoub. First-fit coloring of bounded tolerance graphs. Manuscript.

[9] S.-J. Kim, A. Kostochka, and K. Nakprasit. On the chromatic number of intersection graphs of convex sets in the plane. *Electr. J. Comb.*, 11(1):paper 52, 2004.

[10] A. Kostochka and J. Nešetřil. Coloring relatives of intervals on the plane, I: chromatic number versus girth. *Europ. J. Combin.*, 19:103–110, 1998.

Deliverables

We will have a weekly seminar on coloring problems for geometric structures (twice per week in the first semester). In the first year of the project, we plan to organize a workshop on on-line algorithms for younger researchers from all interested sites. We will also organize annual meetings devoted to the topic of our work package, gathering all contributors.

Justification of the budget

The budget is planned in the same amounts as it was in the outline proposal. The salary costs are planed for 1 senior researcher (J. Grytczuk part-time 0.5), 2 postdocs (Tomasz Krawczyk, Piotr Micek, each 0.5), 3 PhD students (part-time 0.75 each), and stipends for 3 students. We plan travel costs for all members of the research team to travel to conferences, meetings, tutorials, visits, etc. (including the networking within GraDR). Some consumables and necessary overhead are also complied.

Related projects

The current research of Jarosław Grytczuk is supported by institutional funding from the Polish Ministry of Science and Higher Education through the project *Algorithmic problems in combinatorics on words* (grant N N206 257035; individual project). He is also conducting two PhD projects sponsored by the same institution, namely *Coloring distance graphs on the integers* (grant N N201 271335) and *Game coloring of graphs* (grant N201 2128 33).

The Polish Ministry of Science and Higher Education also supports the team project *On-line algorithms and combinatorial games* (grant N N206 492338) led by Piotr Micek. Team members Tomasz Krawczyk and Bartosz Walczak are contributing.

Overview of other ongoing international scientific relationships

Jarosław Grytczuk collaborates frequently with Noga Alon (Tel Aviv University), Boštjan Brešar (University of Maribor), Hal Kierstead (Arizona State University), Sandi Klavžar (University of Ljubljana), and Xuding Zhu (Natinal Sun Yat-sen University). The covered topics include nonrepetitive colorings of graphs, game chromatic number, and on-line Ramsey theory.